Comput Manag Sci
https://doi.org/10.1007/s10287-018-0323-z

ORIGINAL PAPER

Calibration of one-factor and two-factor Hull–White models using swaptions

Vincenzo Russo[1] · **Gabriele Torri**[2,3]

Received: 31 October 2017 / Accepted: 5 June 2018

Abstract In this paper, we analize a novel approach for calibrating the one-factor and the two-factor Hull–White models using swaptions under a market-consistent framework. The technique is based on the pricing formulas for coupon bond options and swaptions proposed by Russo and Fabozzi (J Fixed Income 25:76–82, 2016b; J Fixed Income 27:30–36, 2017b). Under this approach, the volatility of the coupon bond is derived as a function of the stochastic durations. Consequently, the price of coupon bond options and swaptions can be calculated by simply applying standard no-arbitrage pricing theory given the equivalence between the price of a coupon bond option and the price of the corresponding swaption. This approach can be adopted to calibrate parameters of the one-factor and the two-factor Hull–White models using swaptions quoted in the market. It represents an alternative with respect to the existing approaches proposed in the literature and currently used by practitioners. Numerical analyses are provided in order to highlight the quality of the calibration results in comparison with existing models, addressing some computational issues related to the optimization model. In particular, calibration results and sensitivities are provided for the one- and the two-factor models using market data from 2011 to 2016. Finally, an out-of-sample analysis is performed in order to test the ability of the model in fitting swaption prices different from those used in the calibration process.

✉ Gabriele Torri
gabriele.torri@unibg.it

Vincenzo Russo
russovincent@gmail.com

1 Group Risk Management, Assicurazioni Generali S.p.A., Milan, Italy

2 Department of Management, Economics and Quantitative Methods, University of Bergamo, Bergamo, Italy

3 Department of Finance, VŠB-TU Ostrava, Ostrava, Czech Republic

Published online: 19 June 2018

Keywords One-factor Hull–White model · Two-factor Hull–White model · Calibration · Swaption · Coupon bond option

1 Introduction

Interest rate stochastic models are widely used by practitioners for the evaluation of fixed-income instruments. In this context, one- and two-factor short-rate models are the most widely used in interest rate modeling.

Among the one-factor models, the more popular models include the Hull–White model (HW1) proposed in Hull and White (1990), the Black–Karasinski model proposed in Black and Karasinski (1991), and the CIR++ model proposed in Brigo and Mercurio (2006). Recently, squared-Gaussian term structure models have received increased attention in the literature where one-factor models have been proposed by Jamshidian (1995), Pellser (1996) and Russo and Fabozzi (2016a).

Concerning the two-factor models, relevant two-factor models are the two-factor Hull–White model (HW2) proposed in Hull and White (1994b) and the extension of the Longstaff–Schwartz model[1] proposed by Brigo and Mercurio (2006) to fit the term structure of interest rates (CIR2++).

All those models are based on an assumed dynamics in the continuously compounded short-rate. Such models are able to generate yield curves of various realistic forms where the parameters of the models can be estimated quite easily from market data.

However, all the one-factor models have some unrealistic properties. They are not able to generate all the yield curve shapes observed in practice. For example, the Hull–White and the CIR++ models can only produce an increasing curve, a decreasing curve, and a curve with a small hump. Therefore, these models do not allow the so-called *twists* of the term structure of interest rates, where yield curve changes with short-maturity yields and long-maturity yields move in opposite directions. A further critical point is that changes over infinitesimal time periods of any two interest rate dependent variables will be perfectly correlated. This is, for example, the case for any two bond prices or any two yields. This is due to the fact that all unexpected changes are proportional to the shock to the short-rate. It is thus clear that the one-factor diffusion models may very well be too simple to provide a reasonable fit of both the cross-section and time-series dynamics of bond prices.

Instead, two-factor models are more flexible and should be able to generate additional yield curve shapes and yield curve movements relative to the one-factor models. Furthermore, two-factor models are featured by a non-perfect correlations between different interest rate dependent variables.

Despite the wide class of models available in the literature and despite the drawback related to the one-factor models, one-factor and two-factor Hull–White models are among the most common, since they guarantee a good trade-off between analytical tractability and accuracy of the results. One of the interesting features of these models is that they are able to fit the term structure of interest rates. Both models

[1] Longstaff and Schwartz (1992).

admit negative interest rates but this feature is not totally undesirable as, in the recent years and for some currencies, debt instruments are experiencing or have experienced negative interest rates for the short-term and medium-term sectors of the yield curve. Furthermore, the two-factor Hull–White model is featured by a realistic correlation structure between different rates.

When these types of models are used for pricing purposes, they need to be calibrated in a consistent manner using financial instruments quoted in the market. Calibrated models can then be used to evaluate more complex derivatives and structured products. Interest rate models are typically calibrated using caps, floors or swaptions, as these derivatives are among the most liquid instruments traded in the market. In recent years, market practices are moving towards the use of swaptions more than caps and floors, as they contain information on the correlation between different maturities of the interest rates curve. In fact, swaptions are able to capture the negative correlation between stochastic factors in multi-factors interest rate models such as the two-factor Hull–White model.

Under the HW1 the HW2 models, closed-form pricing formulas for zero-coupon bond options, caps and floors are available. Instead, exact pricing formulas for European swaptions and European coupon bond options are not available. Consequently, semi-analytic formulas or numerical techniques as binomial/trinomial trees have to be considered for pricing and calibration purposes.[2]

In this paper, we address the problem related to the evaluation of swaptions for calibrating both the HW1 and the HW2 models under a market-consistent setting. In particular, we use the methodology recently introduced by Russo and Fabozzi (2016b, 2017b). Assuming that the forward price of a coupon bond is a martingale under the forward risk-neutral measure, the proposed approach involves deriving the volatility of the coupon bond as a function of the stochastic durations calculated in the case of, respectively, the HW1 and the HW2 models. Once the volatility function is defined, the price of a coupon bond option can be derived by simply applying standard no-arbitrage pricing theory. Given the equivalence between the price of a coupon bond option and the price of the corresponding swaption, this model can be adopted to calibrate parameters of the HW1 and the HW2 models using swaptions quoted in the market. The advantage of this approach is that, relying on the stochastic duration, it allows us to obtain a convenient formula for the pricing of swaption that requires only the computation of one integral to obtain the volatility of the coupon bond.

We extend the results obtained in Russo and Fabozzi (2016b, 2017b) by analysing the empirical properties of the proposed models and by comparing the proposed models with existing techniques currently used by practitioners: the Jamshidian's approach (Jamshidian 1989) for the one-factor model and the approximation proposed by Schrager and Pelsser (2006) for the two-factor model. We perform numerical analyses in order to highlight the quality of the calibration results in comparison with existing models, highlighting some of the computational issues related to the optimization procedure. Moreover, we calibrate and test the goodness of the models also

[2] See Hull and White (1994a, b, 2001).

under a negative interest rates environment; in this context, we use shifted log-normal swaption quotes for calibration purposes as suggested in Russo and Fabozzi (2017a).

The paper is structured as follows: in Sect. 2 we describe the framework related respectively to one-factor and two-factor Hull–White model, in Sect. 3 we show the pricing of coupon bond options and swaptions, in Sect. 4 we discuss the calibration procedure, in Sect. 5 we present the empirical results. Finally, we provide our conclusion.

2 The framework: one-factor and two-factor Hull–White models

In this section, we describe the one- and the two-factor Hull–White models, following Brigo and Mercurio (2006) and Russo and Fabozzi (2016b, 2017b). We formalize the framework referring to an alternative representation of both the HW1 and the HW2 models in terms of Gaussian processes (with constant parameters) plus a deterministic function.

2.1 One-factor Hull–White model

In the case of the one-factor Hull–White model, the short-rate $r(t)$ at time $t \geq 0$, under the risk-neutral measure, is defined as follows:

$$r(t) = \alpha(t) + x(t), \tag{1}$$

where $x(t)$ is the state variable while $\alpha(t)$ is a deterministic function of time. The variable $x(t)$ is such that

$$dx(t) = -a_x x(t)dt + \sigma_x dW_x(t), \quad x(0) = 0, \tag{2}$$

where a_x and σ_x are model parameters while $dW_x(t)$ is a Brownian motion. Under the HW1 model, the function $\alpha(t)$ is calculated as follows:

$$\alpha(t) = f^M(0,t) + \frac{\sigma_x^2}{2a_x^2}\left(1 - e^{-a_x t}\right)^2. \tag{3}$$

The price of a zero-coupon bond at time t with maturity in $T > t$ can be expressed analytically:

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \frac{P_x(0,t)}{P_x(0,T)} P_x(t,T), \tag{4}$$

where $P^M(0,t)$ is the market price of a zero-coupon bond quoted at $t = 0$ and maturity at t and $P_x(t,T)$ is such that

$$P_x(t,T) = G_x(t,T)e^{-H_x(t,T)x(t)}. \tag{5}$$

Furthermore, we define the following quantities,

$$G_x(t,T) = \exp\left\{ -\frac{\sigma_x^2}{2a_x^2}\left[H_x(t,T) - (T-t)\right] - \frac{\sigma_x^2}{4a_x}H_x(t,T)^2 \right\}, \tag{6}$$

$$H_x(t,T) = \frac{1}{a_x}\left[1 - e^{-a_x(T-t)}\right]. \tag{7}$$

The price of a zero-coupon bond with maturity T satisfies the following stochastic differential equation,

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - \sigma_x D_{P_x}(t,T)dW_x(t). \tag{8}$$

The quantity $D_{P_x}(t,T)$ is the stochastic duration of the zero-coupon bond such that (see Munk 1999):

$$D_{P_x}(t,T) = -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial x(t)} = H_x(t,T). \tag{9}$$

Let's consider a coupon bond with cash flows payments C_i at time T_i such that:

$$\begin{aligned} C_i &= K\tau(T_{i-1},T_i) \quad \text{for } i = 1,2,\ldots,n-1, \\ C_i &= K\tau(T_{i-1},T_i) + 1 \quad \text{for } i = n, \end{aligned}$$

where K is the coupon rate and $\tau(T_{i-1},T_i)$ denotes the time measure between T_{i-1} and T_i computed as a fraction of the year.

Denoting by $B(t,T_n)$ the spot price at time $t \geq 0$ of a coupon bond with maturity T_n, we have that

$$B(t,T_n) = \sum_{i=1}^{n} C_i P(t,T_i). \tag{10}$$

Applying Ito's lemma, we assume that, under the risk-neutral measure, the market value of a coupon bond at time t evolves according to the following stochastic dynamic:

$$\frac{dB(t,T_n)}{B(t,T_n)} = r(t)dt - \sigma_x D_{B_x}(t,T_n)dW_x(t). \tag{11}$$

The quantity $D_{B_x}(t,T_n)$ is the stochastic durations of the coupon bond such that

$$D_{B_x}(t,T_n) = -\frac{1}{B(t,T_n)}\frac{\partial B(t,T_n)}{\partial x(t)} = \frac{\sum_{i=1}^{n} C_i P(t,T_i)H_x(t,T_i)}{\sum_{i=1}^{n} C_i P(t,T_i)}. \tag{12}$$

2.2 Two-factor Hull–White model

Under the risk-neutral measure, the short-rate $r(t)$ at time $t \geq 0$ is defined as follows:

$$r(t) = \alpha(t) + x(t) + y(t), \tag{13}$$

where $x(t)$ and $y(t)$ are state variables while $\alpha(t)$ is a deterministic function of time. The variables $x(t)$ and $y(t)$ are such that

$$dx(t) = -a_x x(t)dt + \sigma_x dW_x(t), \quad x(0) = 0, \tag{14}$$

$$dy(t) = -a_y y(t)dt + \sigma_y dW_y(t), \quad y(0) = 0, \tag{15}$$

where a_x, a_y, σ_x and σ_y are model parameters while $dW_x(t)$ and $dW_y(t)$ are correlated Brownian motions such that

$$dW_x(t)dW_y(t) = \rho dt, \tag{16}$$

with $-1 \leq \rho \leq 1$. Under the HW2 model, the function $\alpha(t)$ is calculated as follows:

$$\alpha(t) = f^M(0,t) + \frac{\sigma_x^2}{2a_x^2}\left(1 - e^{-a_x t}\right)^2 + \frac{\sigma_y^2}{2a_y^2}\left(1 - e^{-a_y t}\right)^2 + \rho\frac{\sigma_x\sigma_y}{a_x a_y}\left(1 - e^{-a_x t}\right)\left(1 - e^{-a_y t}\right). \tag{17}$$

The price of a zero-coupon bond at time t with maturity in $T > t$ can be expressed analytically as follows,

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)}\frac{P_{xy}(0,t)}{P_{xy}(0,T)}P_{xy}(t,T), \tag{18}$$

where $P_{xy}(t,T)$ is such that

$$P_{xy}(t,T) = P_x(t,T)P_y(t,T)C(t,T), \tag{19}$$

and,

$$P_x(t,T) = G_x(t,T)e^{-H_x(t,T)x(t)}, \tag{20}$$

$$P_y(t,T) = G_y(t,T)e^{-H_y(t,T)y(t)}. \tag{21}$$

The quantity $C(t,T)$ represents the correlation-related component of the price and it is formulated as follows,

$$C(t,T) = \exp\left\{\frac{\rho\sigma_x\sigma_y}{a_x a_y}\left[(T-t) - H_x(t,T) - H_y(t,T) + \frac{1}{a_x + a_y}\left(1 - e^{-(a_x+a_y)(T-t)}\right)\right]\right\}. \tag{22}$$

Furthermore, we define the following quantities,

$$G_x(t,T) = \exp\left\{ -\frac{\sigma_x^2}{2a_x^2}\big[H_x(t,T) - (T-t)\big] - \frac{\sigma_x^2}{4a_x} H_x(t,T)^2 \right\}, \tag{23}$$

$$G_y(t,T) = \exp\left\{ -\frac{\sigma_y^2}{2a_y^2}\big[H_y(t,T) - (T-t)\big] - \frac{\sigma_y^2}{4a_y} H_y(t,T)^2 \right\}, \tag{24}$$

$$H_x(t,T) = \frac{1}{a_x}\left[1 - e^{-a_x(T-t)}\right], \tag{25}$$

$$H_y(t,T) = \frac{1}{a_y}\left[1 - e^{-a_y(T-t)}\right]. \tag{26}$$

The price of a zero-coupon bond with maturity T satisfies the following stochastic differential equation,

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - \sigma_x D_{P_x}(t,T)dW_x(t) - \sigma_y D_{P_y}(t,T)dW_y(t). \tag{27}$$

The quantities $D_{P_x}(t,T)$ and $D_{P_y}(t,T)$ are stochastic durations of the zero-coupon bond with respect to the factors x and y respectively

$$D_{P_x}(t,T) = -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial x(t)} = H_x(t,T), \tag{28}$$

$$D_{P_y}(t,T) = -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial y(t)} = H_y(t,T). \tag{29}$$

Applying Ito's lemma, we assume that under the risk-neutral measure the market value of a coupon bond at time t evolves according to the following stochastic dynamic,

$$\frac{dB(t,T_n)}{B(t,T_n)} = r(t)dt - \sigma_x D_{B_x}(t,T_n)dW_x(t) - \sigma_y D_{B_y}(t,T_n)dW_y(t). \tag{30}$$

The quantities $D_{B_x}(t,T_n)$ and $D_{B_y}(t,T_n)$ are the stochastic durations of the coupon bond with respect to the factors x and y respectively,

$$D_{B_x}(t,T_n) = -\frac{1}{B(t,T_n)}\frac{\partial B(t,T_n)}{\partial x(t)} = \frac{\sum_{i=1}^n C_i P(t,T_i) H_x(t,T_i)}{\sum_{i=1}^n C_i P(t,T_i)}, \tag{31}$$

$$D_{B_y}(t,T_n) = -\frac{1}{B(t,T_n)}\frac{\partial B(t,T_n)}{\partial y(t)} = \frac{\sum_{i=1}^n C_i P(t,T_i) H_y(t,T_i)}{\sum_{i=1}^n C_i P(t,T_i)}. \tag{32}$$

3 Pricing coupon bond options and swaptions

Given the spot price at time t of a coupon bond that starts at T_0 with maturity T_n, we define its forward price as,

$$B(t, T_0, T_n) = \frac{B(t, T_n)}{P(t, T_0)}, \tag{33}$$

with $t < T_0 < T_n$. We assume the coupon bond forward price is a martingale under the T-forward measure. Consequently, the ratio between the coupon bond price and the zero-coupon bond price is a martingale under both HW1 and HW2 models.

3.1 Volatility function under the one-factor Hull–White model

In this section, following Russo and Fabozzi (2016b) we show that it is possible to derive the volatility function to be used for coupon bond option and swaption pricing. By application of Ito's lemma we obtain that

$$\frac{dB(t, T_0, T_n)}{B(t, T_0, T_n)} = -\sigma_x D_{B_x}(t, T_0, T_n) dW_x^T(t), \tag{34}$$

where $dW_x^T(t)$ is a Brownian motion under the T-forward measure.

The quantity $D_{B_x}(t, T_0, T_n)$ is defined as the forward stochastic duration of the coupon bond. After simple calculations we obtain,

$$\begin{aligned} D_{B_x}(t, T_0, T_n) &= D_{B_x}(t, T_n) - D_{P_x}(t, T_0) \\ &= \frac{\sum_{i=1}^n C_i P(t, T_i)\big[H_x(t, T_i) - H_x(t, T_0)\big]}{\sum_{i=1}^n C_i P(t, T_i)}. \end{aligned} \tag{35}$$

Given the process above, the variance of the forward price of the coupon bond is

$$\sigma_B(t, T_0, T_n)^2 = \sigma_x^2 D_{B_x}(t, T_0, T_n)^2. \tag{36}$$

In order to provide the pricing function, we need to derive the variance of the coupon bond price under the T-forward risk-adjusted measure,

$$\Sigma_B(t, T_0, T_n)^2 = \int_t^{T_0} \sigma_B(u, T_0, T_n)^2 du. \tag{37}$$

Consequently, we find that the volatility of the coupon bond is

$$\Sigma_B(t, T_0, T_n) = \sqrt{\int_t^{T_0} \sigma_B(u, T_0, T_n)^2 du}. \tag{38}$$

The integral can be solved using a numerical method in order to calculate the volatility of the coupon bond under the proposed model.

3.2 Volatility function under the two-factor Hull–White model

Following Russo and Fabozzi (2017b), by application of the Ito's lemma, we obtain that

$$\frac{dB(t,T_0,T_n)}{B(t,T_0,T_n)} = -\sigma_x D_{B_x}(t,T_0,T_n)dW_x^T(t) - \sigma_y D_{B_y}(t,T_0,T_n)dW_y^T(t),$$

where $dW_x^T(t)$ and $dW_y^T(t)$ are Brownian motions under the T-forward measure.

The quantities $D_{B_x}(t,T_0,T_n)$ and $D_{B_x}(t,T_0,T_n)$ are defined as forward stochastic durations of the coupon bond with respect to the factors x and y respectively. After simple calculations we obtain,

$$\begin{aligned} D_{B_x}(t,T_0,T_n) &= D_{B_x}(t,T_n) - D_{P_x}(t,T_0) \\ &= \frac{\sum_{i=1}^n C_i P(t,T_i)\big[H_x(t,T_i) - H_x(t,T_0)\big]}{\sum_{i=1}^n C_i P(t,T_i)}, \end{aligned} \tag{39}$$

$$\begin{aligned} D_{B_y}(t,T_0,T_n) &= D_{B_y}(t,T_n) - D_{P_y}(t,T_0) \\ &= \frac{\sum_{i=1}^n C_i P(t,T_i)\big[H_y(t,T_i) - H_y(t,T_0)\big]}{\sum_{i=1}^n C_i P(t,T_i)}. \end{aligned} \tag{40}$$

Given the process above, the variance of the forward price of the coupon bond is,

$$\begin{aligned} \sigma_B(t,T_0,T_n)^2 = \sigma_x^2 D_{B_x}(t,T_0,T_n)^2 + \sigma_y^2 D_{B_y}(t,T_0,T_n)^2 \\ + 2\rho\sigma_x\sigma_y D_{B_x}(t,T_0,T_n) D_{B_y}(t,T_0,T_n). \end{aligned}$$

In order to provide the pricing function, we need to derive the variance of the coupon bond price under the T-forward risk-adjusted measure,

$$\Sigma_B(t,T_0,T_n)^2 = \int_t^{T_0} \sigma_B(u,T_0,T_n)^2 du. \tag{41}$$

Consequently, we find that the volatility of the coupon bond is

$$\Sigma_B(t,T_0,T_n) = \sqrt{\int_t^{T_0} \sigma_B(u,T_0,T_n)^2 du}. \tag{42}$$

Also in this case, a numerical integration method is required to calculate the volatility of the coupon bond under the proposed model.

3.3 Closed formula for coupon bond options and swaptions

Consider the value at time t of an option written on a coupon bond that pays fixed annual coupons. The maturity of the option is $T_0 > t$ while the strike price is X. The underlying coupon bond has maturity at time T_n with cash flows $C_1, C_2, \ldots, C_i, \ldots, C_n$ paid at future dates $T_1, T_2, \ldots, T_i, \ldots, T_n$. It is worth noting that, for option pricing purposes, we consider only the bond's cash flows paid after the option's maturity. Consequently, we neglect all the bond's payments between t and T_0. We assume that $B(t, T_n)$ is log-normally distributed under the T-forward risk-adjusted measure. We define as $\mathbb{E}^T$ the expectation under the T-forward risk-adjusted measure denoted by $\mathcal{M}^T$ (the probability measure that is defined by the Radon–Nikodym derivative) and $\mathscr{F}_t$ as the sigma-field generated up to time t.

As coupon bond prices are assumed to be log-normal, we can price options on coupon bonds explicitly following standard option pricing techniques. The results of the previous sections allow us to explicitly calculate the price of a European coupon bond option when the coupon bond price is log-normal and interest rates are stochastic and evolve according to theHW1 or HW2 processes. According to option pricing theory, the arbitrage-free price for a coupon bond call option (CBC) is,

$$CBC(t, T_0, T_n, K, X) = P(t, T_0)\mathbb{E}^T\left[\left(B(T_0, T_n) - X\right)^+ \middle| \mathscr{F}_t\right]. \tag{43}$$

Consequently, we have that

$$CBC(t, T_0, T_n, K, X) = B(t, T_n)\Phi(d_1) - XP(t, T_0)\Phi(d_2), \tag{44}$$

where Φ denotes the cumulative distribution function of the standard Gaussian distribution with

$$d_1 = \frac{\log\left[\frac{B(t,T_n)}{P(t,T_0)}\frac{1}{X}\right] + \frac{1}{2}\Sigma_B(t, T_0, T_n)^2}{\Sigma_B(t, T_0, T_n)}, \tag{45}$$

and

$$d_2 = \frac{\log\left[\frac{B(t,T_n)}{P(t,T_0)}\frac{1}{X}\right] - \frac{1}{2}\Sigma_B(t, T_0, T_n)^2}{\Sigma_B(t, T_0, T_n)}. \tag{46}$$

The corresponding coupon bond put option (CBP) can be obtained by simply applying the put-call parity.

The proposed model can be used also to price swaptions since, from a pricing perspective, swaptions are equivalent to coupon bond options. In particular, a European payer (receiver) swaption can be shown to be equivalent to a European put (call) option on a coupon bond with unitary strike. Denoting respectively by $PSwpt(t, T_0, T_n, K)$ and $RSwpt(t, T_0, T_n, K)$, the price of a payer and a receiver swaption with strike K

and maturity T_0 (both written on an interest rate swap with issue date T_0 and maturity T_n), it follows that

$$CBC(t, T_0, T_n, K, 1) = RSwpt(t, T_0, T_n, K), \tag{47}$$

and

$$CBP(t, T_0, T_n, K, 1) = PSwpt(t, T_0, T_n, K). \tag{48}$$

Finally, swaptions can be evaluated as follows:

$$RSwpt(t, T_0, T_n, K) = B(t, T_n)\Phi(d_1) - P(t, T_0)\Phi(d_2), \tag{49}$$

and

$$PSwpt(t, T_0, T_n, K) = P(t, T_0)\Phi(-d_2) - B(t, T_n)\Phi(-d_1), \tag{50}$$

where

$$d_1 = \frac{\log\left[\frac{B(t,T_n)}{P(t,T_0)}\right] + \frac{1}{2}\Sigma_B(t, T_0, T_n)^2}{\Sigma_B(t, T_0, T_n)}, \tag{51}$$

$$d_2 = \frac{\log\left[\frac{B(t,T_n)}{P(t,T_0)}\right] - \frac{1}{2}\Sigma_B(t, T_0, T_n)^2}{\Sigma_B(t, T_0, T_n)}. \tag{52}$$

4 Model calibration

Using the closed formulas derived in Sect. 3, we can calibrate Hull–White models using prices of swaptions prevailing in the market. In particular, the objective of the calibration process is to choose the model parameters in such a way that the model prices are consistent with swaptions quoted by the market. The solution is found by means of a numeric optimization procedure so as to minimize the square root of the sum of the squares of the relative differences between market and model swaption prices,

$$\arg\min_{\beta} \sqrt{\sum_{i=1}^{N} \left(\frac{Swpt_i - Swpt_i^M}{Swpt_i^M}\right)^2}, \tag{53}$$

where $Swpt_i^M$ is the value of the swaption quoted by the market and $Swpt_i$ represents the swaption's theoretical price under the HW1 or HW2 models. The number of calibrated instruments is N, while β is the vector of parameters. The HW1 model requires the calibration of 2 parameters (a and σ) while the HW2 model has 5 parameters (a_x, a_y, σ_x, σ_y and ρ).

We calibrate all the model parameters at the same time in order to match market prices. It is worth to highlight that it is one of the possible approaches that can be adopted. In fact, some authors propose to estimate the mean reversion parameter using historical data rather than calibrate it in the context of the optimization procedure described above. Others estimate it in two separate steps. For instance, Schlenkrich (2012) proposes to calibrate the mean reversion parameter using Bermudan swaptions, and the volatility parameter using European swaptions.

In the calibration process we use theoretical swaption prices ($Swpt_i$) calculated under the approach proposed by Russo and Fabozzi (2016b) for the HW1 model, and Russo and Fabozzi (2017b) for the HW2 model. The numerical integrals in Eqs. (37) and (41) are solved by discretizing the time at one month intervals.[3] The optimization problem is not convex, and we search for a local optimum with a simulated annealing algorithm (Ingber 1996). Concerning the HW1 model, the convergence is rather fast and stable, while for the HW2 model we test multiple starting points to avoid local minima. In line with the common practices, we perform the calibration of co-maturity swaptions (see Sect. 5.1). Still, this practice may induce some overfitting and parameter instability, especially for the two-factor model. In Sect. 5.2 we repeat the calibration process by considering a different calibration set that includes a grid of swaptions characterized by different combinations of tenor and maturity.

In addition, we compare the calibration results with the results obtained using respectively the Jamshidian's approach and the Schrager–Pelsser approximation.

The Jamshidian's approach is usually adopted by practitioners to calculate prices for coupon bond options and requires the calculation of a zero-coupon bond option price for each of the payment dates of the coupon bond after the option's expiration date. Although this method provides analytical pricing for swaptions, the formula is not explicit in the model parameters. In fact, the critical value, for which the price of the coupon-bearing bond equals the strike price of the option on the bond at option maturity, has to be computed numerically.

Under the HW2 model, a well-known technique for the evaluation of European swaptions is the one proposed by Brigo and Mercurio (2006). This solution involves a semi-analytical formula that requires the numerical evaluation of an integral. However, this approach poses some issues in practice since the integral does not have clear boundaries and its evaluation requires to truncate the integration region. Instead, we consider the Schrager–Pelsser approximation[4] that is computationally more efficient in comparison with the solution proposed by Brigo–Mercurio. However, a drawback of the Schrager–Pelsser method is that the approximation error is smaller for shorter tenors and maturity options and grows marginally for swaptions with higher tenors and maturity.

Alternatively, the models could be calibrated using numerical schemes based on binomial/trinomial trees, as proposed in Hull and White (1994a, b) and further discussed in Hull and White (2001).

[3] Preliminary analyses show that the resolution of the discretization do not influence significantly the calibration. Results are available upon request and are not reported for brevity.

[4] This approach is implemented in Di Francesco (2012).

Table 1 Calibrated parameters for the one-factor Hull–White model

Year	Model	a	σ
2011	Jamshidian	0.1412	0.0181
	Russo–Fabozzi	0.1298	0.0155
2012	Jamshidian	0.0803	0.0115
	Russo–Fabozzi	0.1173	0.0120
2013	Jamshidian	0.1050	0.0142
	Russo–Fabozzi	0.1983	0.0185
2014	Jamshidian	0.0320	0.0076
	Russo–Fabozzi	0.0004	0.0057
2015	Jamshidian	0.0220	0.0080
	Russo–Fabozzi	0.0012	0.0065
2016	Jamshidian	0.0304	0.0085
	Russo–Fabozzi	0.0005	0.0065

5 Numerical results

5.1 Calibration results

The calibration of both the HW1 and the HW2 models is performed using EUR swaption prices obtained from Bloomberg for the last business day in 2011, 2012, 2013, 2014, 2015 and 2016. In particular, we have considered at-the-money (ATM) co-terminal swaptions. This is a common practice adopted to calibrate interest rate models, in part due to hedging reasons. We highlight that for years 2011–2014 Black/log-normal swaptions volatilities have been considered for calibration purposes while for years 2015–2016 shifted log-normal swaption volatilities have been used. It is needed in order to address the issue featuring the Black model under negative rate environments.[5]

Tables 1 and 2 report the parameters of the one-factor Hull–White model (HW1) and the two-factor Hull–White model (HW2). The HW1 model has been calibrated using the Jamshidian decomposition (J) and the Russo–Fabozzi approach (RF). The HW2 model has been calibrated using the Schrager–Pelsser approximation (SP) and the Russo–Fabozzi approach. In order to evaluate the quality fitting, Table 3 reports the Root Mean Square Percentage Error (RMSPE) calculated using market and model prices under the two approaches analyzed, computed as in Eq. (53).

Concerning the HW1 model (Table 1), the comparison of the calibrated parameters presented in Table 1 shows rather similar values between the two approaches analyzed before 2013, while since 2014 the Russo–Fabozzi approach selects smaller values for the parameter a. Concerning the calibration error, The Russo–Fabozzi model shows in all the cases smaller RMSPE (see Table 3, cf. columns 2 and 3).

Regarding the HW2 model calibration (Table 2), we note that the values of the parameters of the process y (a_y and σ_y), are generally smaller than parameters of

[5] See Russo and Fabozzi (2017a) for further details about models calibration practices under negative rates.

Table 2 Calibrated parameters for the two-factor Hull–White model

Year	Model	a_x	σ_x	a_y	σ_y	ρ
2011	Schrager–Pelsser	1.2056	0.0124	0.1685	0.0216	– 1.0000
	Russo–Fabozzi	0.2789	0.0991	0.0179	0.0229	– 0.9999
2012	Schrager–Pelsser	0.6065	0.0296	0.1296	0.0172	– 0.6188
	Russo–Fabozzi	0.9987	0.0387	0.0229	0.0096	– 0.9142
2013	Schrager–Pelsser	0.7617	0.0483	0.1621	0.0259	– 1.0000
	Russo–Fabozzi	0.9998	0.0560	0.0208	0.0116	– 0.9441
2014	Schrager–Pelsser	0.4617	0.0223	0.0623	0.0128	– 1.0000
	Russo–Fabozzi	0.5248	0.0640	0.0149	0.0086	– 0.8306
2015	Schrager–Pelsser	0.5322	0.0133	0.0338	0.0100	– 0.9380
	Russo–Fabozzi	0.3124	0.0510	0.0174	0.0140	– 0.9818
2016	Schrager–Pelsser	0.2770	0.0118	0.0551	0.0136	– 1.0000
	Russo–Fabozzi	0.0756	0.2278	0.0220	0.0246	– 0.9975

Table 3 Model calibration: RMSPE for HW1 and HW2 models

Year	HW1 J	HW1 RF	HW2 SP	HW2 RF
2011	0.173	0.085	0.167	0.085
2012	0.030	0.024	0.025	0.025
2013	0.123	0.046	0.036	0.039
2014	0.173	0.089	0.072	0.058
2015	0.077	0.053	0.015	0.035
2016	0.078	0.048	0.015	0.039

the process x (a_x and σ_x) for both the calibration procedures (Russo–Fabozzi and Schrager–Pelsser). This result is similar to the one obtained by Brigo and Mercurio (2006). Concerning the correlation parameter, as the calibration process is performed using swaptions, the correlation parameters should be different from -1. However, using the Schrager–Pelsser approach, the correlation parameter ρ is equal to -1 in four of the six trading days considered in the numerical analysis. In contrast, using the Russo–Fabozzi approach, ρ is always strictly greater than -1, implying that the model is more capable to capture the correlation between interest rates with different maturities. As far as the calibration error, we see in Table 3 that the two-factor model, having more parameters, performs overall better than the one-factor model showing smaller RMSPE (cf. columns 3 and 5). Comparing the two calibration techniques (Russo–Fabozzi and Schager–Pelsser, cf. columns 4 and 5), we see that none of the approaches dominate the other, with the Schager–Pelsser performing better in 2013, 2015 and 2016, and the Russo–Fabozzi in 2011, 2012 and 2014. the RMSPE are very similar, with the Russo–Fabozzi approach method showing slightly better performances.

In general, the calibration procedure under the Russo–Fabozzi approach seems to perform well in comparison to state-of-art techniques used by practitioners. With respect to such practices, the Russo–Fabozzi approach shows similar calibration errors, good analytical properties and, in the case of the HW2 model, the ability to estimate correlation coefficients different than -1.

5.2 Out-of-sample analysis

The calibration performed in Sect. 5.1 has been done on 10-years co-maturity swaptions, as commonly done by practitioners. Such choice, although motivated by hedging considerations, may lead to sub-optimal calibration results due to inefficient use of data (only a small sub-set of swaptions quoted in the market are considered), and overfitting, especially concerning the two-factor model that has a larger number of parameters to calibrate.

Here, we test the out-of-sample performances of the HW1 and HW2 models, calibrated with the Russo–Fabozzi approach, by pricing a set of swaptions different from those used in the calibration process. In particular, we evaluate a set of 170 swaptions with maturity between 1 and 10 years and tenor between 1 month and 30 years. We price the instruments using Eqs. (49)–(52). We then compare model and market prices using the RMSPE computed as in (53). Notice that, by introducing a common calibration technique for the HW1 and the HW2 models, we are able to compare the two models for pricing purposes, limiting the influence of calibration procedures.

We also test a calibration procedure that considers not only co-maturity swaptions, but instead uses a set of 45 swaptions that span all the combinations of maturity and tenor. This allows us to test the potential overfitting of the model to the calibration set and to test how well the HW1 and HW2 models can fit the market swaption prices.

Table 4 reports the RMSPE for the HW1 model, calibrated on either 10-years co-maturity swaptions, and the grid of 45 swaptions. For both procedures we report the RMSPE for the calibration set and the one for the entire set of swaptions. Table 5 reports the same data for the HW2 model. We see that the good in-sample performances of the HW1 model calibrated on 10-years co-maturities swaptions (column 2 of Table 4) are not completely reflected by the pricing of the other swaptions in the market, that present higher errors (column 3). The calibration performed on a set of swaptions larger than the 10-years co-maturity allows to reduce the out-of-sample pricing error (column 5), although the in-sample fit is not as good as in the previous case, denoting that the parametrization of the HW1 model is too narrow to fit the entire set of swaptions. Table 5 shows the results for the HW2 model. As we highlighted in the previous section, the in-sample fit on co-maturity swaptions is marginally better than the HW1 (column 2). Here we see that this improvement comes at the cost of worse pricing of swaptions outside the calibration set (see column 3), suggesting the presence of overfitting. The results for the alternative calibration set (columns 4 and 5) show instead much better results compared to the HW1 model, thanks to the richer parametrization.

These results suggest that the HW2 model allows to better fit the price of swaption across all the spectrum of tenures and maturities, but that the calibration set has to be chosen carefully to avoid overfitting. Practitioners interested in hedging applications that require a particularly accurate fit of a certain set of swaptions, may choose a

Table 4 Model calibration: RMSPE for the HW1 model calibrated using 10-years co-maturity swaptions and a grid of 45 swaptions with different tenor and maturity

Year	Co-maturity swaptions		Grid of swaptions	
	Calib. set	All	Calib. set	All
2011	0.0850	0.3922	0.1396	0.1472
2012	0.0240	0.5636	0.1689	0.1796
2013	0.0458	0.8050	0.2297	0.2137
2014	0.0890	0.4665	0.2866	0.3227
2015	0.0534	0.4837	0.3188	0.3169
2016	0.0476	0.8129	0.4970	0.4721

Table 5 Model calibration: RMSPE for the HW2 model calibrated using 10-years co-maturity swaptions and a grid of 45 swaptions with different tenor and maturity

Year	Co-maturity swaptions		Grid of swaptions	
	Calib. set	All	Calib. set	All
2011	0.0852	0.3979	0.0765	0.0893
2012	0.0252	0.7352	0.0730	0.0637
2013	0.0387	0.3343	0.0891	0.0796
2014	0.0577	2.1734	0.2846	0.3373
2015	0.0350	2.2916	0.0528	0.0493
2016	0.0394	3.3505	0.0729	0.0605

different weighting scheme for the objective function in the optimization, in order to give more importance to certain assets.

5.3 Sensitivity analysis

We provide here a brief analysis in order to study the sensitivity of swaption prices to changes in parameters in the HW1 and HW2 models.

Figure 1 reports the effects on receiver swaptions prices of changes in the mean-reversion parameter a (top panels) and interest rate volatility σ (bottom panels) under the HW1. We have performed the analysis using swaptions with different tenors and two different maturities, 0.5 and 5 years. The x axes report the tenor of the receiver swaption while the y axes reports its price. The baseline values of the parameters are $a = 0.1$ and $\sigma = 0.02$. Looking at the upper panels we see that, the higher the parameter a, the lower the price of the swaption. Moreover, the effect of changes in a are particularly relevant for swaptions with long tenor, while for shorter tenors the effect is limited, especially for swaptions with short maturity (Fig. 1, top-left panel). Changes in σ instead have an opposite effect: higher levels of volatility lead to higher prices, and the effect is consistent across maturities and tenors.

Figure 2 reports the sensitivity analysis for the HW2 model. The baseline values for the parameters are $a_x = 0.3$, $a_y = 0.1$, $\sigma_x = 0.02$, $\sigma_y = 0.02$ and $\rho = -0.8$. We change a_y, σ_y and ρ with the values specified in Fig. 2. Focusing on the correlation coefficient ρ, we see that values closer to -1 are consistent with lower swaptions

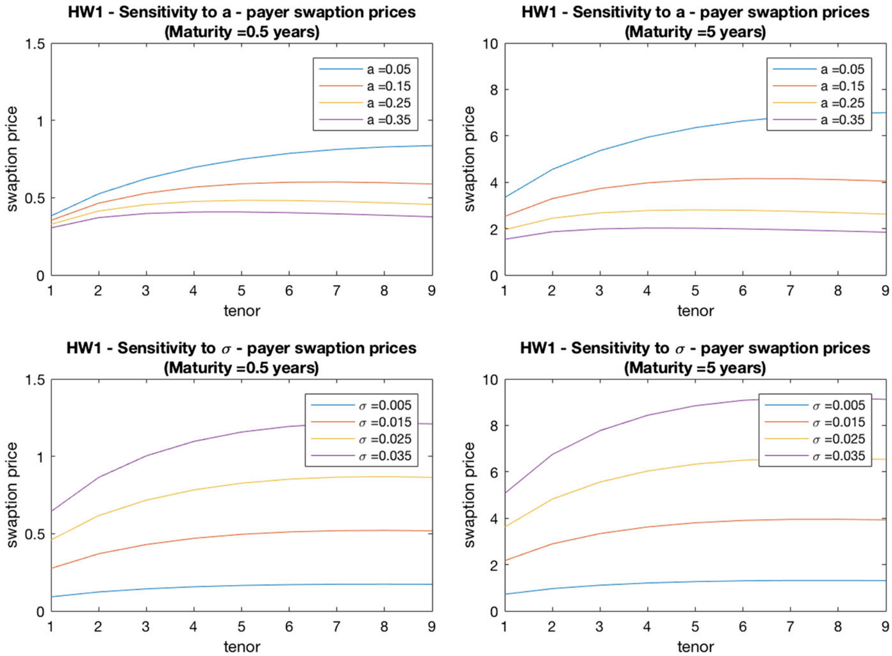

Fig. 1 HW1 model - Receiver swaption prices for different levels of σ and a

prices. The changes in prices are similar for different tenors (note the parallel shifts in the graphs in bottom panels of Fig. 2), and the changes are stronger for shorter maturities.

Overall, the two-factor Hull–White model allows a more complete parametrization of the system, with a larger set of parameters that allows to fit accurately swaption prices.

In relation to our optimization procedure, we also test the sensitivity of the RMSPE with respect to the parameters of the HW1 and HW2 models, computing the error for a grid of suitably chosen parameter values. Similarly to Sect. 5.2, we consider both the RMSPE computed on 10-years co-maturity swaptions and on the entire grid of 170 swaptions with tenors ranging from 1 month to 30 years and maturity from 1 to 10 years.

The results for the HW1 model are reported in Fig. 3, where the color represents the value of the RMSPE. We can see that, in the case of the 10-years co-maturity swaptions (left panel), the local minimum is located in a "long valley", making the optimization problem ill-posed and the solution likely to be unstable. Instead, the RMSPE surface for the entire set of swaptions (right panel) appears to have a better defined optimum. Figure 4 reports the sensitivity results for the HW2 model by showing some slices of the RMSPE surface. In particular, we report the values for different combinations of the parameters $a_1, a_2, \sigma_1, \sigma_2$ and ρ; each panel shows the surface for two parameters, while the others are assigned the optimal value computed for the year 2011. The optimization problem is clearly more challenging for the HW2 compared to the HW1

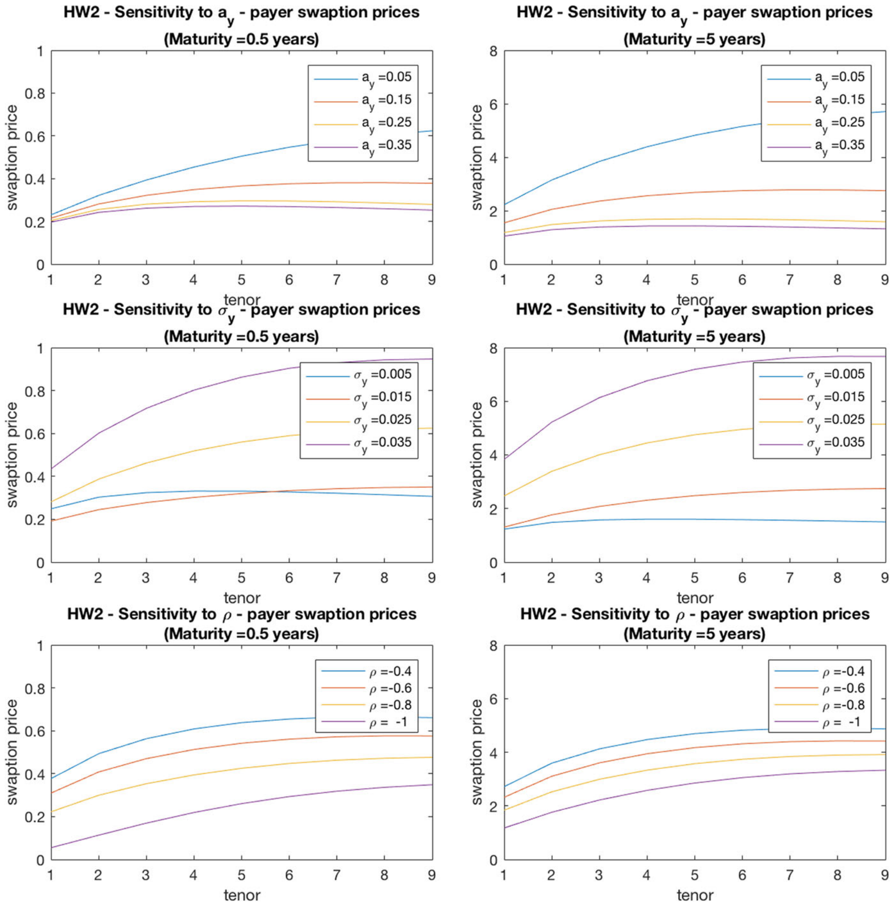

Fig. 2 HW2 model—receiver swaption prices for different levels of σ and a

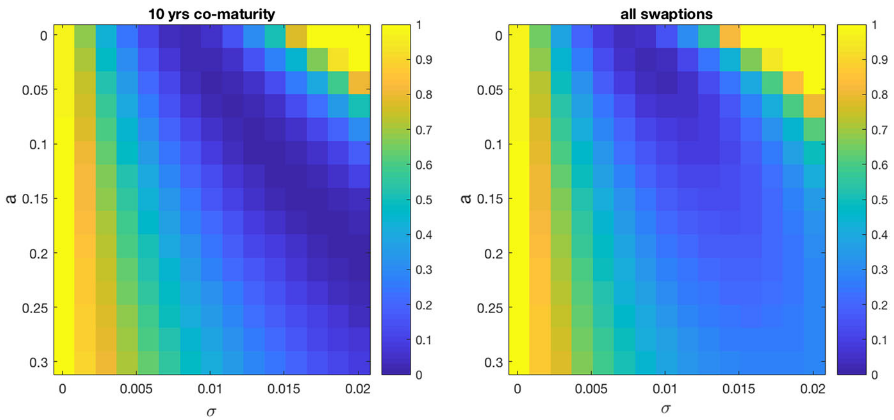

Fig. 3 HW1 model—RMSPE for different model parameters

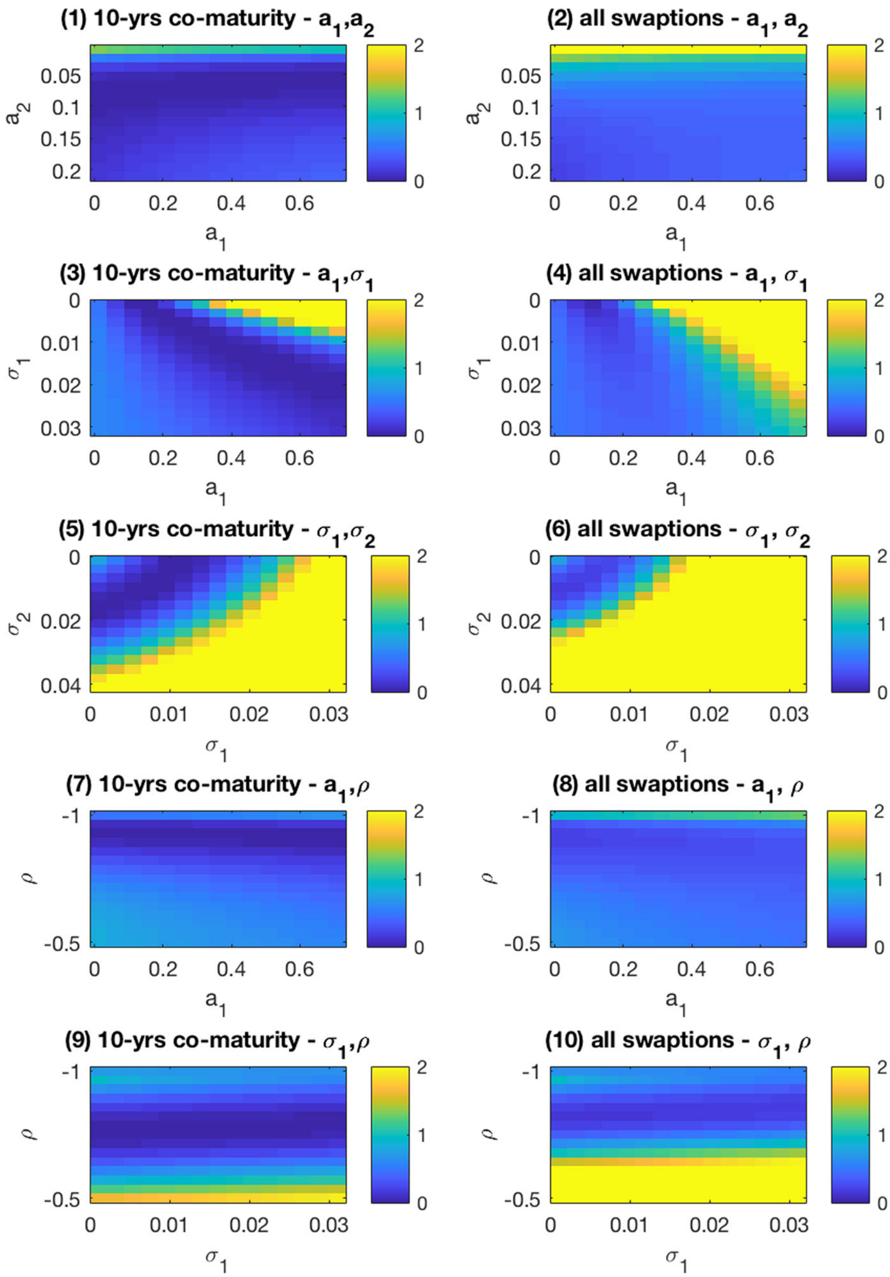

Fig. 4 HW2 model—RMSPE for different model parameters

model due to the higher dimensionality and the complexity of the objective function. Still, we see that, also in this case, the surfaces for the 10-years co-maturity swaptions set seem to present less clearly defined optima due to the presence of valleys that may compromise the stability of the result. This is consistent with the results in Sect. 5.2, where the calibration on co-maturity swaptions gave bad out-of-sample performances, suggesting to pay particular care to the selection of the calibration set.

6 Conclusion

In this paper, we address the problem related to the evaluation of swaptions for calibrating both the one-factor and the two-factor Hull–White models under a market-consistent setting. In particular, we use the methodology recently introduced by Russo and Fabozzi (2016b, 2017b). Assuming that the forward price of a coupon bond is a martingale under the forward risk-neutral measure, the proposed approach involves deriving the volatility of the coupon bond as a function of the stochastic durations calculated in the case of, respectively, one-factor and two-factor Hull–White models. Once the volatility function is defined, the price of a coupon bond option can be derived by simply applying standard no-arbitrage pricing theory. Given the equivalence between the price of a coupon bond option and the price of the corresponding swaption, this model can be adopted to calibrate parameters of the one-factor and the two-factor Hull–White models using swaptions quoted in the market. The advantage of this approach is that, relying on the stochastic duration, it allows us to obtain a convenient formula for the pricing of swaption that requires only the computation of one integral to obtain the volatility of the coupon bond.

We have extended the results obtained in Russo and Fabozzi (2016b, 2017b) by analyzing the empirical properties of the proposed models and by comparing the proposed models with existing techniques currently used by practitioners: the Jamshidian's approach (Jamshidian 1989) for the one-factor model and the approximation proposed by Schrager and Pelsser (2006) for the two-factor model. Numerical analyses have been performed in order to highlight the quality of the calibration results in comparison with existing models. Moreover, following Russo and Fabozzi (2017a), we have calibrated and tested the goodness of the models under a negative interest rates environment using shifted log-normal swaption quotes for calibration purposes. In particular, some computational issues have been addressed to solve the optimization model implemented to calibrate parameters. In general, the calibration procedure under the Russo–Fabozzi approach seems to perform well in comparison to state-of-art techniques used by practitioners. With respect to such practices, the Russo–Fabozzi approach shows similar calibration errors, good analytical properties and, in the case of the HW2 model, the ability to estimate correlation coefficients different than -1. In addition, we performed an analysis aimed at studying the sensitivity of the solution to changes in model parameters and an out-of-sample analysis in which we used the calibrated model to price swaptions outside the calibration set. The sensitivity analysis and the out-of-sample evaluation of the calibrated model suggest that the calibration on co-maturity swaptions, although commonly used in the industry, may lead to sub-optimal calibration, due to over fitting and inefficient use of the data, we recommend therefore to consider a larger set of swaptions for the calibration to improve the stability of the results.

We can conclude that the Russo–Fabozzi approach presents several advantages compared to the other calibration techniques. In fact, it relies on a largely analytical pricing formula (requiring only the numerical computation of one integral), it can be applied to both the one- and the two-factor Hull–White models and it does not require approximations.

Acknowledgements Gabriele Torri acknowledges the support of the Czech Science Foundation (GACR) under Project 15-23699S and SP2017/32, an SGS research project of VSB-TU Ostrava.

Disclaimer Vincenzo Russo and not his employer is solely responsible for any errors.

References

Black F, Karasinski P (1991) Bond and option pricing when short rates are lognormal. Financ Anal J 47:52–59

Brigo D, Mercurio F (2006) Interest rate models: theory and practice, 2nd edn. Springer, Berlin

Di Francesco, M (2012) A general Gaussian interest rate model consistent with the current term structure. ISRN Probab Stat 2012, Article ID 673607, 16 pages

Hull J, White A (1990) Pricing interest rate derivative securities. Rev Financ Stud 3:573–592

Hull J, White A (1994a) Numerical procedure for implementing term structure models I: single-factor models. J Deriv 2:7–16

Hull J, White A (1994b) Numerical procedure for implementing term structure models II: two factor models. J Deriv 2:37–47

Hull J, White A (2001) The general Hull–White model and supercalibration. Financ Anal J 57(6):34–43

Ingber L (1996) Adaptive simulated annealing (ASA): lessons learned. Control Cybern 25:33–54

Jamshidian F (1989) An exact bond option formula. J Finance 44:205–209

Jamshidian F (1995) A simple class of square root interest rate models. Appl Math Finance 2:61–72

Longstaff FA, Schwartz ES (1992) Interest rate volatility and the term structure: a two-factor general equilibrium model. J Finance 47(4):1259–1282

Munk C (1999) Stochastic duration and fast coupon bond option pricing in multi-factor models. Rev Deriv Res 3(2):157–181

Pellser A (1996) A tractable interest rate model that guarantees positive interest rates. Rev Deriv Res 1:269–284

Russo V, Fabozzi FJ (2016a) A one-factor shifted squared gaussian term structure model for interest rate modeling. J Fixed Income 25:36–45

Russo V, Fabozzi FJ (2016b) Pricing coupon bond options and swaptions under the one-factor Hull-White model. J Fixed Income 25:76–82

Russo V, Fabozzi FJ (2017a) Calibrating short interest rate models in negative rate environments. J Deriv 24:80–92

Russo V, Fabozzi FJ (2017b) Pricing coupon bond options and swaptions under the two-factor Hull–White model. J Fixed Income 27:30–36

Schlenkrich S (2012) Efficient calibration of the Hull–White model. Optim Control Appl Methods 33(3):352–362

Schrager DF, Pelsser A (2006) Pricing swaptions and coupon bond options in affine term structure models? Math Finance 16:673–694